

Assignment V: MTH 213, Fall 2017

Ayman Badawi

QUESTION 1. Let $n \in N^*$. Prove that $nCk = nC(n-k)$ for every k, $0 \leq k \leq n$. Use direct prove (hint: Write down the formula for each and just stare!, so now we know $20C3 = 20C17, 61C40 = 61C21$, and so on...)

QUESTION 2. Let $n \in N^*$. Prove that $(n+1)C(k+1) = nC(k+1) + nCk$ for every k, $0 \leq k \leq n-1$. Use direct prove (hint: Write down the formula for each. Now some how make the denominator of the right hand side = the denominator of left hand side... and you should get it. This fact is used when we constructed Pascal Triangle)

QUESTION 3. Use Math. Induction to prove that $nC0 + nC1 + \cdots + nCn = 2^n$ for every $n \in N^*$. ([Hint: In the last step, you need to use the fact from Question 2.)

QUESTION 4. Give me a direct proof of the fact: $nC0 + nC1 + \cdots + nCn = 2^n$ for every $n \in N^*$.

QUESTION 5. Use Math. Induction to prove that $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for every $n \in N^*$.

QUESTION 6. Let $x_1 = 4, x_{n+1} = \sqrt{3 + 4x_n}$. Use Math. Induction to prove that $x_n \leq 5$ for every $n \geq 1$.

QUESTION 7. Write down T or F. If you select F, then show me by example why it is F

(i) $\exists x \in N^*$ and $\exists y \in Z$ such that $x + y = 0$.

(ii) $\exists x \in N^*$ such that $x + y = 0 \forall y \in Z$.

(iii) $\forall y \in Q^* \exists x \in Q$ such that $xy = 2$ (you read this as for every y... there exists x ...)

(iv) $\exists! x \in N$ such that $yx = 4y \ \forall \ y \in R$

(v) $\exists! x \in N$ such that $yx = 4y \ \forall \ y \in Q^*$

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: `abadawi@aus.edu, www.ayman-badawi.com`

Hanin Alrais

Q1. $n \in N^*$
Prove that $nCk = nC(n-k)$
for every k, $0 \leq k \leq n$.

$$nCk = \frac{n!}{k! \cdot (n-k)!}$$

$$nC(n-k) = \frac{n!}{(n-k)!\,(n-(n-k))!}$$

$$= \frac{n!}{(n-k)! \cdot (+k)!}$$

Hence, by direct proof we see that the formula for both is =

Q2. $n \in N^*$
Prove that $(n+1)C(k+1) = nC(k+1) + nCk$ for every k,
$0 \leq k \leq n-1$

① $(n+1)C(k+1) = \dfrac{(n+1)!}{(k+1)! \cdot ((n+1)-(k+1))!}$

② $nC(k+1) + nCk = \dfrac{n!}{(k+1)! \cdot (n-(k+1))!} + \dfrac{n!}{k!\,(n-k)!}$

①=② $\dfrac{(n+1)!}{(k+1)!\,[n+1-k-1]!} = \dfrac{n!}{(k+1)!\,(n-k-1)!} + \dfrac{n!}{k!\,(n-k)!}$

remove $n!$ and $k!$ from both sides

$$\frac{(n+1)}{(k+1)(n-k)!} = \frac{1}{(k+1)[n-k-1]!} + \frac{1}{(n-k)!}$$

remove $[n-k-1]!$

$$\frac{(n+1)}{(k+1)(n-k)} = \frac{1}{(k+1)} + \frac{1}{(n-k)}$$

$$\frac{(n+1)}{(k+1)(n-k)} = \frac{(n-k)+(k+1)}{(k+1)(n-k)} \checkmark$$

Q3. Use Induction to prove
$nC0 + nC1 + \ldots + nCn = 2^n$
for every $n \in N^*$

① Prove it for $n=1$

$1C0 + 1C1 = 2^1$ ✓

② Assume it is true for $n = k \geq 1$

$kC0 + kC1 + \ldots + kCk = 2^k$ ✓

③ Prove it for $n = k+1$
prove $(k+1)Ci = 2^{(k+1)}$

$\Rightarrow (k+1)C0 + (k+1)C1 + \ldots + (k+1)C(k+1) = 2^{k+1}$

Use fact from Q2:
so $\binom{k+1}{0} = 1$,

$\binom{k+1}{1} = \binom{k}{1} + \binom{k}{0}$,

$\binom{k+1}{2} = \binom{k}{2} + \binom{k}{1}$,

$\binom{k+1}{3} = \binom{k}{3} + \binom{k}{2}$, ...

$\binom{k+1}{4} = \binom{k}{4} + \binom{k}{3}$, ...

..., $\binom{k+1}{k} = \binom{k}{k} + \binom{k}{k-1}$

$\binom{k+1}{k+1} = 1$

Here, we conclude that:
$\binom{k}{0}$ and $\binom{k}{k}$ appear once,
however the others appear twice
the sequnce continues.

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(iv) $\forall y \in Q^* \ \exists x \in Q$ s.t.

$xy = 2$

True $\quad 4 \times \frac{1}{2} = 2$

$\frac{5}{2} \times \frac{4}{5} = 2$

(v) $\exists! \, x \in N$ s.t. $yx = 4y \ \forall y \in R$

False $\quad$ assume $y = 0$, then x can be any number

(vi) $\exists! \, x \in N$ s.t. $yx = 4y \ \forall y \in Q^*$

True $\quad x = 4$

Q4. Give the direct proof of the fact : $nC0 + nC1 + \dots nCn = 2^n$ for every $n \in N^*$

For every real number x we know that

$(X + 1)^n = nC0X^n + nC1X^{n-1} + \dots + nC(n-1)X + nCn.$

So let $X = 1$ in the equation above. We have

$2^n = nC0 + nC1 + \dots + nC(n-1) + nCn$

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Q5. Use Induction to prove $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for every $n \in \mathbb{N}^*$

① Prove it for $n=1$

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{(1+1)} = \frac{1}{2} \checkmark$$

② Assume it is true for $n = k \geq 1$

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{(k+1)} \text{ is true} \checkmark$$

③ Prove it for $n = k+1$
we need to show that

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{(k+1)+1}$$

$$= \sum_{i=1}^{k} \frac{k}{(k+1)} + \frac{1}{(k+1)(k+1+1)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2} = \frac{k+1}{(k+1)+1} \checkmark$$

Q6. $x_1 = 4$, $x_{n+1} = \sqrt{3+4x_n}$
Use Math. Induction to prove that $x_n \leqslant 5$ for every $n \geq 1$

① Prove for $n=1$

$$x_{1+1} = \sqrt{3+4x_1} = \sqrt{3+4(4)}$$

$$= \sqrt{19} \leqslant 5 \checkmark$$

② Assume it is true for $n = k \geq 1$

$$x_k = \sqrt{3+4x_{k-1}} \leqslant 5$$

③ Prove it for $n = k+1$,

$$x_{k+1} = \sqrt{3+4x_k} \leqslant \sqrt{3+4(5)}$$

$$\leqslant \sqrt{23} < 5$$

Q7. (i) $\exists x \in \mathbb{N}^*$ and $\exists y \in \mathbb{Z}$ s.t. $x+y=0$

True $1+(-1)=0$
$2+(-2)=0$
⋮

(ii) $\exists x \in \mathbb{N}^*$ s.t. $x+y=0$ $\forall y \in \mathbb{Z}$

False $1+(-2) = -1 \neq 0$
$1+2=3 \neq 0$

(iii) $\exists x \in \mathbb{Z}^*$ s.t $x+y=0$ $\forall y \in \mathbb{Z}$

False $2+0=2 \neq 0$

∴ we can say that:

$$2\left[\binom{k}{0}\binom{k}{1}\binom{k}{2}\cdots\binom{k}{k}\right]+2-2$$

* we multiplied by 2 because they each occurred twice.
* we subtracted by 2 because $\binom{k}{0}$ and $\binom{k}{k}$ occurred once, not twice, * we add 2 because $\binom{k+1}{0}=1$ and $\binom{k+1}{k+1}=1$.

from the assumption 2 we can see that.

$$2\underbrace{\left[\binom{k}{0}\binom{k}{1}\binom{k}{2}\cdots\binom{k}{k}\right]}\cancel{+2-2}$$

$$2\cdot 2^{k}=2^{(k+1)}$$